169. **On Lacunary Trigonometric Series. II**

By Shigeru TAKAHASHI

Department of Mathematics, Kanazawa University, Kanazawa

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§ 1. Introduction. In [3] we have proved

**Theorem A.** Let \( \{n_k\} \) be a sequence of positive integers and \( \{a_k\} \) a sequence of non-negative real numbers for which the conditions

\[
\begin{align*}
n_{k+1} &> n_k(1+c k^{-s}), \quad k=1, 2, \cdots, \\
A_N &= (2^{-1} \sum_{k=1}^{N} a_k^{1/2} )^{1/2} \to +\infty, \quad \text{as } N \to +\infty,
\end{align*}
\]

and

\[
\begin{align*}
a_N &= o(A_N^{1-s}), \quad \text{as } N \to +\infty,
\end{align*}
\]

are satisfied, where \( c \) and \( \alpha \) are any given constants such that

\[
c > 0 \quad \text{and} \quad 0 < \alpha \leq 1/2.
\]

Then we have, for all \( x \),

\[
\begin{align*}
limit_{N \to \infty} \left| \left\{ t; t \in E, \sum_{k=1}^{N} a_k \cos 2\pi n_k (t + \alpha_k) \leq x A_N \right\} \right| / |E| \\
= (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2)du, \quad (*)
\end{align*}
\]

where \( E \subset [0, 1] \) is any given set of positive measure and \( \{a_k\} \) any given sequence of real numbers.

This theorem was first proved by R. Salem and A. Zygmund in case of \( \alpha = 0 \), where \( \{n_k\} \) satisfies the so-called Hadamard’s gap condition (cf. [4], (5.5), pp. 264–268). In that case they also remarked that under the hypothesis (1.2) the condition (1.3) is necessary for the validity of (1.5) (cf. [4], (5.27), pp. 268–269).

Further, in [2] P. Erdős has pointed out that for every positive constant \( c \) there exists a sequence of positive integers \( \{n_k\} \) such that \( n_{k+1} > n_k(1 + c k^{-1/2}) \), \( k \geq 1 \), and (1.5) is not true for \( a_k = 1, k \geq 1 \), and \( E = [0, 1] \). But I could not follow his argument on the example.

The purpose of the present note is to prove the following

**Theorem B.** For any given constants \( c > 0 \) and \( 0 < \alpha \leq 1/2 \), there exist sequences of positive integers \( \{n_k\} \) and non-negative real numbers \( \{a_k\} \) for which the conditions (1.1), (1.2) and

\[
a_N = O(A_N^{1-s}), \quad \text{as } N \to +\infty,
\]

are satisfied, but (1.5) is not true for \( E = [0, 1] \) and \( \alpha_k = 0, k \geq 1 \).

The above theorem shows that in Theorem A the condition (1.3) is

\[*\] \( |E| \) denotes the Lebesgue measure of a set \( E \).
indispensable for the validity of (1.5). In §§ 3–5 we prove Theorem B for $0 < \alpha \leq 1/2$.

§ 2. Some lemmas. i. In this section let $\{X_k(\omega)\}$ be a sequence of independent random variables on some probability space $(\Omega, \mathcal{F}, P)$ with vanishing mean values and finite variances. Putting $E(X_k^2) = \sigma_k^2$ and $s_m^2 = \sum_{k=1}^{m} \sigma_k^2$, the theorem of Lindeberg reads as follows:

Theorem. (Cf. [1] pp. 56–57.) Under the hypotheses
\begin{equation}
\tag{2.1} s_m \to +\infty \quad \text{and} \quad \sigma_m = o(s_m), \quad \text{as} \quad m \to +\infty,
\end{equation}
a necessary and sufficient condition for the validity of the relation
\begin{equation}
\tag{2.2} \lim_{m \to +\infty} P\{s_m^{-1} \sum_{k=1}^{m} X_k(\omega) \leq x\} = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2) du
\end{equation}
for all $x$ is that, for any given $\varepsilon > 0$,
\begin{equation}
\tag{2.3} \lim_{m \to +\infty} \sum_{k=1}^{m} P\{X_k(\omega) > \varepsilon s_m\} = 0.
\end{equation}

From the above theorem the following lemma is easily seen.

Lemma 1. Under the hypotheses (2.1), the relation (2.2) implies that, for any given $\varepsilon > 0$,
\begin{equation}
\tag{2.3} \lim_{m \to +\infty} \sum_{k=1}^{m} P\{X_k(\omega) > \varepsilon s_m\} = 0.
\end{equation}

ii. Lemma 2. Let $m$ and $l$ be any given positive integers. Then there exists a positive constant $c_0$, not depending on $m$ and $l$, such that
\begin{equation}
|\{t; t \in [0, 1], |\sum_{j=0}^{l} \cos 2\pi m(j+1)t| > (l+1)/3\}| \geq 2c_0 l^{-1}.
\end{equation}

Proof. This can be easily seen from the relation
\begin{equation}
\sum_{j=0}^{l} \cos 2\pi m(j+1)t = \frac{\sin 2\pi m(l+3/2)t}{2 \sin \pi mt} - 1/2,
\end{equation}
provided if $\pi mt \neq 0$.

§ 3. Construction of sequences. In the following let $c > 0$ and $0 < \alpha \leq 1/2$ be given constants in Theorem B. First let us put
\begin{equation}
\tag{3.1}
\begin{cases}
p(j) = \lceil j^{1/\alpha} \rceil, \\
l(j) = \text{Min} \{\lceil p(j)/c \rceil, p(j+1) - p(j) - 1\}, \\
j_0 = \text{Min} \{j; l(j) \geq 1\}.
\end{cases}
\end{equation}

Since $p(j+1) - p(j) \sim \alpha^{-1} j^{(1-\alpha)/\alpha}$ and $p(j) \sim j$, as $j \to +\infty$, we have
\begin{equation}
\tag{3.2} l(j) \sim \beta(\alpha) j, \quad \text{as} \quad j \to +\infty,
\end{equation}
where
\begin{equation}
\tag{3.3} \beta(\alpha) = \begin{cases}
1/c, & \text{if } 0 < \alpha < 1/2, \\
\text{Min} (2, 1/c), & \text{if } \alpha = 1/2.
\end{cases}
\end{equation}

Next we put
\begin{equation}
n_1 = 1 \quad \text{and} \quad n_{k+1} = \lceil n_k (1 + ck^{-\alpha}) + 1 \rceil, \quad \text{for} \quad k+1 < p(j_0).
\end{equation}

If $n_{p(j)}$, $j \geq j_0$, is defined, then we put
\begin{itemize}
\item[(*)] $[x]$ denotes the integral part of $x$.
\item[(**)]] $f(j) \sim g(j)$, as $j \to +\infty$, means that $f(j)/g(j) \to 1$, as $j \to +\infty$.
\end{itemize}
Further we put, for $j \geq j_0$,
\begin{equation}
\tag{3.4}
n_{p(j)} = 2q(j),
\end{equation}
where
\begin{equation}
\tag{3.5}
q(j) = \begin{cases}
\min \{m : 2^m/n_{p(j-1)} > 1 + e\{p(j-1) - 1\}^{-\alpha} \}
& \text{if } j = j_0, \\
\min \{m : 2^m/n_{p(j-1)} > 1 + e\{p(j-1) - 1\}^{-\alpha} \}
& \text{and } 2^m > n_{p(j-1)} j_0^2, \\
\end{cases}
\end{equation}
Then it is clear that the sequence \{n_k\} satisfies (1.1).

On the other hand we define \{a_k\} as follows:
\begin{equation}
\tag{3.6}
a_k = \begin{cases}
1 & \text{if } p(j) \leq k \leq p(j) + l(j), \text{ for some } j \geq j_0, \\
1/k & \text{if otherwise.}
\end{cases}
\end{equation}
Then we have, by (3.6) and (3.2),
\begin{equation}
\tag{3.7}
A_{p(m)} + l(m) = 2^{-1} \sum_{j=j_0}^m \sum_{i=0}^{l(j)} \cos \{2\pi 2^q(j) (1 + 1)t\} + O(1), \text{ as } m \to +\infty.
\end{equation}
Since $p(j) \to j_0$, as $j \to +\infty$, this sequence \{a_k\} satisfies both of the conditions (1.2) and (1.6).

Further, if we put
\begin{equation}
\tag{4.1}
S_n(t) = \sum_{k=1}^N a_k \cos 2\pi n_k t, \quad \text{then we have, by the definitions of } \{n_k\} \text{ and } \{a_k\}, \text{ uniformly in } t,
\end{equation}
\begin{equation}
\tag{4.2}
S_{p(m) + l(m)}(t) = 2^{-1} \sum_{j=j_0}^m \sum_{i=0}^{l(j)} \cos \{2\pi 2^q(j) (1 + 1)t\} + O(1), \text{ as } m \to +\infty.
\end{equation}

§ 4. Independent functions. Let $x_k(t)$ be the $k$-th digit of the infinite dyadic expansion of $t$, $0 \leq t \leq 1$, that is,
\begin{equation}
\tag{4.3}
t = \sum_{k=-\infty}^{\infty} x_k(t) 2^{-k}, \quad (x_k(t) = 0 \text{ or } 1),
\end{equation}
then \{x_k(t)\} is a sequence of independent functions on the interval $[0, 1]$. Putting
\begin{equation}
\tag{4.4}
\eta_j(t) = \sum_{k=-q(j)}^{q(j+1)-1} x_k(t) 2^{-k}, \quad \text{for } j \geq j_0,
\end{equation}
we define
\begin{equation}
\tag{4.5}
\begin{cases}
\mu_j = \int_{0}^{l(j)} \sum_{i=0}^{l(j)} \cos \{2\pi 2^q(j) (1 + 1)t\} \eta_j(t) dt, \\
Y_j(t) = \sum_{i=0}^{l(j)} \cos \{2\pi 2^q(j) (1 + 1)t\} \eta_j(t) - \mu_j, \\
\tau_j^2 = \int_{0}^{1} Y_j(t)^2 dt \quad \text{and } t_j^2 = \sum_{j=j_0}^m \tau_j^2.
\end{cases}
\end{equation}
Then we have, by (4.2) and (3.5),
\begin{equation}
\sup \left| \int_{0}^{l(j)} \sum_{i=0}^{l(j)} \cos \{2\pi 2^q(j) (1 + 1)t\} - \sum_{i=0}^{l(j)} \cos \{2\pi 2^q(j) (1 + 1)\eta_j(t)\} \right|
= O(\sup t \sum_{k=q(j+1)}^{\infty} x_k(t) 2^{-k} \sum_{i=0}^{l(j)} (l + 1))
= O(2^{q(j+1)} j^2 \sum_{k=q(j+1)}^{\infty} 2^{-k})
= O(2^{q(j+1)} j^2) = O(j^{-1}), \quad \text{as } j \to +\infty.
\end{equation}
Therefore, we have
\[(4.4) \quad \sup_t |Y_j(t) - \sum_{l=0}^{l(j)} \cos 2\pi 2^{l(j)}(l+1)t| = O(j^{-1}), \quad \text{as } j \to +\infty,\]
and, by (3.2) and (3.7),
\[(4.5) \quad t_m^2 = 2^{-1} \sum_{j=0}^{m} (l(j)+1)^2 + O(\log m)\]
~ \(A_p^2_p)_{p(m)} \sim \beta(\alpha)m^2/4,\]
\[\text{as } m \to +\infty.\]
Thus we obtain
\[(4.6) \quad t_m \to +\infty \quad \text{and} \quad \tau_m = o(t_m), \quad \text{as } m \to +\infty.\]
Further, we have, by (3.8) and (4.4),
\[(4.7) \quad |S_{p(m)} + l(m)_{l(m)} - \sum_{j=0}^{m} Y_j(t)| = O(\log m)\]
~ \(o(A_p^2_p)_{p(m)}), \text{ uniformly in } t, \text{ as } m \to +\infty.\]
Since (4.5) and (3.2) imply that \(\sqrt{\beta(\alpha)} t_m/5 < [l(m/2)+1]/4, \text{ for } m \geq m_0,\)
we have, by (4.4)
\[\sum_{j=m/2}^{m} |\{t; 0<t \leq 1, |Y_j(t)| > \sqrt{\beta(\alpha)} t_m/5\}|\]
\[\geq \sum_{j=m/2}^{m} |\{t; 0 \leq t \leq 1, \sum_{l=0}^{l(j)} \cos 2\pi 2^{l(j)}(l+1)t > [l(j)+1]/3\}|.\]
and by Lemma 2, we have
\[(4.8) \quad \lim_{m \to \infty} \sum_{j=m/2}^{m} |\{t; 0 \leq t \leq 1, |Y_j(t)| > \sqrt{\beta(\alpha)} t_m/5\}|\]
\[\geq \lim_{m \to \infty} 2c_0 \sum_{j=m/2}^{m} l^{-1}(j) \geq \lim_{m \to \infty} c_0 \beta(\alpha)^{-1} \sum_{j=m/2}^{m} j^{-1} > 0.\]

§ 5. Proof of Theorem B. Suppose that (1.5) holds, that is,
\[(5.1) \quad \lim_{m \to \infty} |\{t; 0 \leq t \leq 1, A_{p(m)} + l(m)_{l(m)} S_{p(m)} + l(m)_{l(m)}(t) \leq x\}|\]
\[= (2\pi)^{-1/2} \int_{-x}^{x} \exp(-u^2/2)du.\]
Then by (4.7) and (4.5), we have
\[(5.2) \quad \lim_{m \to \infty} |\{t; 0 \leq t \leq 1, t_m^{-1} \sum_{j=0}^{m} Y_j(t) \leq x\}|\]
\[= (2\pi)^{-1/2} \int_{-x}^{x} \exp(-u^2/2)du.\]
On the other hand (4.1), (4.2), and (4.3) imply that \(\{Y_j(t)\}\) is a sequence of independent functions with vanishing mean values and finite variances. By (5.2) and (4.6) we can apply Lemma 1 to \(\{Y_j(t)\}\) and obtain
\[\lim_{m \to \infty} \sum_{j=m/2}^{m} |\{t; 0 \leq t \leq 1, |Y_j(t)| > \sqrt{\beta(\alpha)} t_m/5\}| = 0.\]
This contradicts with (4.8).

References

